Introduction to Logarithms

Logarithms are commonly credited to a Scottish mathematician named John Napier who constructed a table of values that allowed multiplications to be performed by addition of the values from the table.

Logarithms are used in many situations such as:

(a) Logarithmic Scales: the most common example of these are pH, sound and earthquake intensity.

(b) Logarithm Laws are used in Psychology, Music and other fields of study

(c) In Mathematical Modelling: Logarithms can be used to assist in determining the equation between variables.

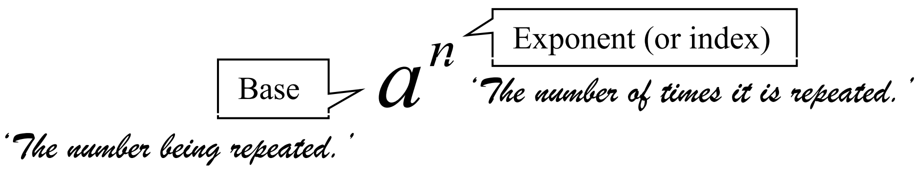
Logarithms were used by most high-school students for calculations prior to scientific calculators being used. This involved using a mathematical table book containing logarithms. Slide rules were also used prior to the introduction of scientific calculators. The design of this device was based on a Logarithmic scale rather than a linear scale.

There is a strong link between numbers written in **exponential form** and **logarithms**, so before starting Logs, let's review some concepts of exponents (Indices) and exponent rules.

The Language of Exponents

The **power** a^n can be written in expanded form as: $a^n = a \times a \times a \times a \times a \ldots\ldots\ldots\ldots \times a$ $\left[\text{for } n \text{ factors}\right]$

The **power** a^n consists of a **base** a and an **exponent** (or index) n.



Examples:

$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

[3] [x■] [5] [=] 243

3W5é 243

Multiplication or division of powers with the **same base** can be simplified using the Product and Quotient Rules.

The Product Rule

$$a^m \times a^n = a^{m+n}$$

When multiplying two powers with the same base, add the exponents.

$$3^2 \times 3^4 = 3^6 = 729$$
$$4^{-1} \times 4^5 \times 4^2 = 4^{-1+5+2} = 4^6 = 4096$$
$$b^2 \times b^7 = b^{2+7} = b^9$$

The Quotient Rule

$$a^m \div a^n = a^{m-n}$$

When dividing two powers with the same base, subtract the exponents.

$$3^7 \div 3^2 = 3^{7-2} = 3^5 = 243$$
$$4^2 \times 4^4 \div 4^3 = 4^{2+4-3} = 4^3 = 64$$
$$e^6 \div e^4 = e^{6-4} = e^2$$

The zero exponent rules can also be used to simplify exponents.

The Zero Exponent Rule

$$a^0 = 1$$

A power with a zero exponent is equal to 1.

$$3^0 = 1$$
$$113750^0 = 1$$
$$e^0 = 1$$
$$(4x)^0 = 1$$

The power rule can help simplify when there is a power to a power.

The Power Rule

$$\left(a^m\right)^n = a^{m \times n}$$

When a power is raised to a power, multiply the exponents.

$$\left(3^2\right)^4 = 3^{2\times4} = 3^8 = 6561$$
$$\left(5^0\right)^6 = 5^{0\times6} = 5^0 = 1$$
$$\left(b^2\right)^7 = b^{2\times7} = b^{14}$$

Two more useful Power Rules are:

$$(ab)^m = a^m b^m \quad \text{or} \quad a^m b^m = (ab)^m$$
$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad \text{or} \quad \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

$$(3a)^3 = 3^3 \times a^3$$
$$\left(5a^2\right)^3 = 5^3 \times a^{2\times3} = 125a^6$$
$$\left(\frac{4}{a}\right)^4 = \frac{4^4}{a^4} = \frac{256}{a}$$

The negative exponent rule is useful when a power with a negative exponent needs to be expressed with a positive exponent.

The Negative Exponent Rule

$$a^{-m} = \frac{1}{a^m} \quad \text{or} \quad \frac{1}{a^{-m}} = a^m$$

Take the reciprocal and change the sign of the exponent.

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$$

$$b^{-7} = \frac{1}{b^7}$$

$$\frac{4}{a^{-3}} = 4 \times a^3 = 4a^3$$

The fraction exponent rule establishes the link between fractional exponents and roots.

The Fractional Exponent Rule

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

for unit fractions, or

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \text{or} \quad \left(\sqrt[n]{a}\right)^m$$

for any fraction

$$3^{\frac{1}{2}} = \sqrt[2]{3} = \sqrt{3}$$

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4$$

$$p^{\frac{3}{4}} = \sqrt[4]{p^3}$$

$$b^{-\frac{1}{4}} = \frac{1}{b^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{b}}$$

Exponent Functions found on a Scientific Calculator

Function	Appearance of Key	Example
Square 3^2	[x^2]	[3] [x^2] [=] 9
Cube 2^3	[x^3]	[2] [x^3] [=] 8
Any exponent 3^5	for W	[3] [$x^{\blacksquare}$] [5] [=] 243 3W5é 243
Square root $\sqrt{16}$	s or s	[$\sqrt{\blacksquare}$] [1] [6] [=] 4
Cube root $\sqrt[3]{343}$	($\sqrt[3]{\blacksquare}$) or S	[SHIFT] [x^3] [3] [4] [3] [=] 7 Generally [SHIFT] [x^3] gives ($\sqrt[3]{\blacksquare}$) or S (although this varies between different makes and models)

Any root $\sqrt[10]{200}$	F or $\boxed{\sqrt[x]{\ }}$	[1] [0] [SHIFT] [7] [2] [0] [0] [=] 1.699 10qW200p 1.699 Generally qf gives F or qW gives $\boxed{\sqrt[x]{\ }}$ (although this varies between different makes and models)

Definition of a Logarithm

"When you think about Logarithms you should think about exponents (indices)"

The number 32 can be written as 2^5.

Q: Does that mean that the Logarithm of 32 is equal to 5?

A: Partially! The logarithm of 32 does equal 5 but only when a base of 2 is used.

As a Logarithm, this can be written as $\log_2 32 = 5$

We know that $216 = 6^3$

the Log (logarithm) of 216 to the base 6 is 3

The log is the exponent (3); the exponent is 3 because the base used was 6.

$216 = 6^3$ is equivalent to writing $\log_6 216 = 3$

$216 = 6^3$ is written in exponential form $\log_6 216 = 3$ is written in logarithmic form

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A few more examples:

$$625 = 5^4 \text{ is equivalent to writing } \log_5 625 = 4$$

$$1\,000\,000 = 10^6 \text{ is equivalent to writing } \log_{10} 1\,000\,000 = 6$$

$$1024 = 2^{10} \text{ is equivalent to writing } \log_2 1024 = 10$$

Integral and fractional exponents are also possible

$$\frac{1}{8} = \frac{1}{2^3} = 2^{-3} \text{ is equivalent to writing } \log_2 \frac{1}{8} = -3 \text{ (alternatively } \log_2 0.125 = -3\text{)}$$

$$5 = \sqrt{25} = 25^{\frac{1}{2}} = 25^{0.5} \text{ is equivalent to writing } \log_{25} 5 = 0.5$$

$$1 = 7^0 \text{ is equivalent to writing } \log_7 1 = 0$$

Generalising all of the above:

> If a number (N) is written as a power with a base (b) and an exponent (e), such as
> $$N = b^e$$
> then $\log_b N = e$.
> The base, b, must be a positive number ($b > 0$) and not equal to 1 ($b \neq 1$).
> The number, N, must also be positive ($N > 0$)

Examples:

Write the following in logarithmic form.

	Exponential Form	Logarithmic Form
(a)	$36 = 6^2$	$\log_6 36 = 2$
(b)	$128 = 2^7$	$\log_2 128 = 7$
(c)	$1 = 9^0$	$\log_9 1 = 0$
(d)	$3 = 9^{\frac{1}{2}}$ ($\sqrt{9}$ is the same as $9^{\frac{1}{2}}$)	$\log_9 3 = \frac{1}{2}$
(e)	$4 = 8^{\frac{2}{3}}$	$\log_8 4 = \frac{2}{3}$
(f)	$a = c^2$	$\log_c a = 2$

The base is 6

The base is 6

$216 = 6^3$ is equivalent to writing $\log_6 216 = 3$

The exponent is 3

The log of the number is 3

Note: The logarithm of 1 to any base is always 0.

$$\log_b 1 = 0$$

Why? Remember the zero exponent rule

$b^0 = 1$ written as $1 = b^0$

In Logarithmic Form becomes

$$\log_b 1 = 0$$

Video 'The Definition of a Log(arithm) 1'

In the next set of questions, the logarithmic form is given and is to be written in exponential form.

Write the following in exponential form.

	Logarithmic Form	Exponential Form
(a)	$\log_5 25 = 2$	$25 = 5^2$
(b)	$\log_{10} 1000 = 3$	$1000 = 10^3$
(c)	$\log_5 a = 3.5$	$a = 5^{3.5}$
(d)	$\log_b 10 = 2$	$10 = b^2$
(e)	$\log_2 9 = k$	$9 = 2^k$

Video 'The Definition of a Log(arithm) 2'

Finding Log Values

Your scientific calculator can find the values of logs. Most scientific calculators have two logarithmic functions; Ordinary Logarithms and Natural Logarithms.

Ordinary Logarithms have a base of 10. Because we use a base 10 number system, it seems straight forward that Logs with a base of 10 are used. The Log key on a scientific calculator has the appearance g.

'When you are calculating the log of a number, you can assume it is with a base of ten unless it is indicated otherwise.'

In ordinary logarithms, when you find the Log of a number, you are finding the exponent when a base of 10 is used.

Find the value of $Log 25$ (meaning $Log_{10} 25$)

On your calculator, the sequence of keys is: [log] [2] [5] [=] 1.397940009

If $Log 25$ = 1.397940009 this can be written in exponential form as $25=10^{1.397940009}$.

Find the value of $Log 14500$ (meaning $Log_{10} 14500$)

On your calculator, the sequence of keys is: [log] [1] [4] [5] [0] [0] [=] 4.161368002

If $Log 14500$ = 4.161368002 this can be written in exponential form as $14500=10^{4.161368002}$.

Natural Logarithms

Natural Logarithms have a base of *e*. You may already know about π (pi) which is a constant used in situations involving circles or cyclic events. The constant *e* is used in situations involving growth and decay such as population growth.

The constant *e* has a value 2.7183 (rounded to 4 decimal places).

The natural log key on a scientific calculator has the appearance [ln].

The natural log of a number can be written as $\ln N$ or $\log_e N$. When you find the natural log of a number, you are finding the exponent when a base of *e* (2.7183) is used.

Find the value of $\ln 25$ (which is equivalent to $\log_e 25$)

On your calculator, the sequence of keys is: [ln] [2] [5] [=] 3.218875825

The $\ln 25 = 3.218875825$ which in exponential form is $25 = e^{3.218875825}$.

Find the value of $\ln 14500$ [or $\log_e 14500$]

On your calculator, the sequence of keys is: [ln] [1] [4] [5] [0] [0] [=] 9.581903928

The $\ln 14500 = 9.581903928$ which in exponential form is $14500 = e^{9.581903928}$.

Video 'Logs - Using your calculator'

It would be unusual to be asked to find a logarithm of a number with a base different to 10 or *e*. If you were, then the section below is relevant.

Other Bases

Finding the value of a logarithm to a base other than 10 or *e* can be performed on you calculator.

However, this requires the use of one of Log Laws. The Log Laws are covered in another section. The Log Law used here is called the Change of Base Law.

This law allows a logarithm with a given base to be changed to a new base, the new base being one that is available on your calculator, that is, base 10 or base *e*.

The Change of Base Law can be stated as:

$$Log_a N = \frac{Log_b N}{Log_b a}$$

If **base 10** is chosen as the new base, then the Law can be written as:

$$Log_a N = \frac{Log_{10} N}{Log_{10} a} = \frac{Log N}{Log a}$$

If **base *e*** is chosen as the new base, then the Law can be written as:

$$Log_a N = \frac{Log_e N}{Log_e a} = \frac{\ln N}{\ln a}$$

For example: Find the value of

$$\log_4 40$$

Using the Change of Base Law:

Using the g key.

$$\log_4 40 = \frac{\log 40}{\log 4} = \frac{1.602059991}{0.602059991} = 2.661 \text{ (to 4 d.p.)}$$

Calculator key sequence: [log] [4] [0] [÷] [log] [4] [=] 2.660964047

Using the h key.

$$\log_4 40 = \frac{\log_e 40}{\log_e 4} = \frac{\ln 40}{\ln 4} = \frac{1.602059991}{0.602059991} = 2.661 \text{ (to 4 d.p.)}$$

Calculator key sequence: [ln] [4] [0] [÷] [ln] [4] [=] 2.660964047

Working in reverse!

If you are presented with a question where you know the ordinary logarithm of the number and you want to find out what the number is, then you will be working in reverse.

Such a question will be present like this one: $\log x = 0.4771$

Remember, if no base is given then the **implied** base is 10. Writing in the base in the question will assist in getting to a solution.

The question can be rewritten as: $\log_{10} x = 0.4771$

Now, using the definition of a logarithm, $x = 10^{0.4771}$

Look closely at your calculator, you will notice that the SHIFT (or 2nd F) of the log key is G or (10^x). The log and its inverse use the same key!

Using the calculator, for example, `qg0.4771=`
The answer 2.999853181 will be displayed. Remember `qg` gives the G or (10^x) function. For all practical purposes the value of *x* is 3.

$$\log_{10} x = 0.4771 \quad \rightarrow \quad x = 3$$

If the question involves natural logs, a question could be written as: $\ln x = 2.14$

Remember, *ln* implies natural logs, so the implied base is *e*, writing in the base in the question will assist in getting to a solution.

The question can be rewritten as: $\log_e x = 2.14$

Now, using the definition of a logarithm, $x = e^{2.14}$

Look closely at your calculator, you will notice that the SHIFT (or 2nd F) of the ln key is H or (e^x). The ln and its inverse use the same key!

Using the calculator, for example, `qh2.14=`
The answer 8.499437629 will be displayed. Remember `qh` gives the H or (e^x) function. For all practical purposes the value of *x* is 8.5 !

$$\log_e x = 2.14 \quad \rightarrow \quad x = 8.5$$

If the question involves a logarithm of any base, a question could be written as: $\log_5 x = 0.6065$ This would be unlikely to occur but if it did it is quite easily solved.

Now, using the definition of a logarithm, $x = 5^{0.6065}$

This time you will use a familiar key; the `f` or `w`.

Using the calculator, for example, `5f0.6065=`
The answer 2.654149047 will be displayed. For practical purposes the value of x is 2.65 depending on the accuracy required.

$$\log_5 x = 0.6065 \quad \rightarrow \quad x = 2.65$$

Video 'Logs in reverse!'

Activity

1. Write the following in logarithmic form.

(a)	$25 = 5^2$	(b)	$1024 = 4^5$
(c)	$11.18 = 5^{1.5}$	(d)	$908.14 = 3^{6.2}$
(e)	$0.015625 = 8^{-2}$	(f)	$49^{0.5} = 7$
(g)	$0.4884 = 6^{-0.4}$	(h)	$0.5 = 4^{-0.5}$
(i)	$7.389 = e^2$	(j)	$1.568 = e^{0.45}$

2. Write the following in exponential form.

(a)	$\log 100 = 2$	(b)	$\log_2 1024 = 10$
(c)	$\log 2.65 = 0.4232$	(d)	$\log_4 100 = 3.322$
(e)	$\ln 1 = 0$	(f)	$\ln 20 = 2.996$
(g)	$\log_{100} 10 = \frac{1}{2}$	(h)	$\ln 5 = 1.6094$

3. Evaluate the following logarithms using your calculator. Answer to 4 decimal places.

(a)	$\log 250$	(b)	$\log 2.95$
(c)	$\log 0.1$	(d)	$\log\left(\frac{3}{8}\right)$
(e)	$\ln 3.05$	(f)	$\ln 1000$
(g)	$\ln 0.025$	(h)	$\ln\left(\frac{1}{9}\right)$
(i)	$\log_4 32$	(j)	$\log_{0.25} 4$

4. Working in reverse, find the value of x.

(a)	$\log x = 0.7782$	(b)	$\log x = 2.8779$
(c)	$\log x = -0.9031$	(d)	$\log(2x-1) = 1.6902$
(e)	$\ln x = 2.72$	(f)	$\ln x = 0.75$
(g)	$\ln x = -0.5$	(h)	$\ln 2x = \frac{3}{4}$
(i)	$\log_2 x = 2.72$	(j)	$\log_5 x = -0.25$

Introduction

Before starting this topic you should review the Definition of a Log topic.

Properties of Logarithms

These are the basics of Logs, based on the definition of Logs.

1.	$\log_a 1 = 0$ $\ln 1 = 0,\ \log 1 = 0$	From exponents, we know that any number raised to the power of 0 is equal to 1, ie: $a^0 = 1$, using the definition of a Log, $\log_a 1 = 0$
2.	$\log_a a = 1$ $\ln e = 1,\ \log 10 = 1$	From exponents, we know that any number raised to the power 1 is itself, ie: $a^1 = a$, using the definition of a Log, $\log_a a = 1$
3.	$\log_a a^n = n$ $\ln e^n = n,\ \log 10^n = n$	From exponents, $a^n = a^n$, using the definition of a Log, $\log_a a^n = n$
4.	$a^{\log_a n} = n$ $e^{\ln x} = x,\ 10^{\log x} = x$	Starting with $\log_a n = \log_a n$, using the definition of a Log, in exponential form: $n = a^{\log_a n}$

The Three Logarithm Laws

The Log Laws help simply Log expressions.

The first law: (Product Rule)

$$\log_b A + \log_b B = \log_b AB$$

Note: all three bases are the same.

Proof:

Let $\log_b A = x$ then $A = b^x$

Let $\log_b B = y$ then $B = b^y$

then $AB = b^x \times b^y = b^{x+y}$ (Index Law 1)

In exponent form: $AB = b^{x+y}$

to Log form: $\log_b AB = x + y$ (using the definition)

$$\log_b AB = \log_b A + \log_b B$$

Examples:

Simplify $\log_2 5 + \log_2 4$

$$\log_2 5 + \log_2 4$$
$$= \log_2 (5 \times 4)$$
$$= \log_2 20$$

Simplify $\log 5 + \log 8 + \log 3$ (with no base mentioned, the inference is that the base is 10)

$$\log 5 + \log 8 + \log 3$$
$$= \log(5 \times 8 \times 3)$$
$$= \log 120$$

Simplify $\log 2x + \log 3y$

$$\log 2x + \log 3y$$
$$= \log(2x \times 3y)$$
$$= \log 6xy$$

Simplify $\ln x + \ln y + \ln 10$ (Remember: the natural logarithm is base *e*)

$$\ln x + \ln y + \ln 10$$
$$= \ln(x \times y \times 10)$$
$$= \ln(10xy)$$

The first law used in reverse is:

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$ and $\log 5 = 0.6990$, evaluate $\log 30$

$$\log 30$$
$$= \log(2 \times 15)$$
$$= \log(2 \times 3 \times 5)$$
$$= \log 2 + \log 3 + \log 5$$
$$= 0.3010 + 0.4771 + 0.6990$$
$$= 1.4771$$

The second law: (Quotient Rule)

$$\log_b A - \log_b B = \log_b \frac{A}{B}$$

Note: all three bases are the same.

Proof:

Let $\log_b A = x$ then $A = b^x$

Let $\log_b B = y$ then $B = b^y$

then $\frac{A}{B} = \frac{b^x}{b^y} = b^{x-y}$ (Index Law 2)

In exponent form: $\frac{A}{B} = b^{x-y}$

to Log form: $\log_b \frac{A}{B} = x - y$ (using the definition)

$$\log_b \frac{A}{B} = \log_b A - \log_b B$$

Examples:

Simplify $\log_2 20 - \log_2 4$

$$\begin{aligned} &\log_2 20 - \log_2 4 \\ &= \log_2\left(\frac{20}{4}\right) \\ &= \log_2 5 \end{aligned}$$

Simply $\log 50 - \log 5 - \log 2$

$$\begin{aligned} &\log 50 - \log 5 - \log 2 \\ &= \log 50 - (\log 5 + \log 2) \\ &= \log 50 - \log 10 \\ &= \log \frac{50}{10} \\ &= \log 5 \end{aligned}$$

Or in reverse:

Given $\log 2 = 0.3010,$, evaluate $\log 50$

$$\begin{aligned} &\log 50 \\ &= \log \frac{100}{2} \\ &= \log 100 - \log 2 \\ &= 2 - \log 2 \quad (\log_{10} 100 = 2 \text{ because } 100 = 10^2) \\ &= 2 - 0.3010 \\ &= 1.699 \end{aligned}$$

The third law: (Power Rule)

$$\log_a x^n = n \log_a x$$

Proof:

$$\begin{aligned} &\text{let } \log_a x = m \\ &\quad \text{then } x = a^m \\ &\qquad x^n = \left(a^m\right)^n \text{ raise both sides to the power of } n \\ &\qquad x^n = a^{mn} \text{ using the power of a power exponent rule} \\ &\log_a x^n = mn \text{ (or } nm\text{) using the definition of a log} \\ &\log_a x^n = n \log_a x \end{aligned}$$

Examples: Evaluate (or simplify)

$\log_3 \frac{1}{3} = \log_3 3^{-1} = -1\log_3 3 = -1$ or $\log_3 \frac{1}{3} = \log_3 1 - \log_3 3 = 0 - 1 = -1$

$\log x^4 = 4 \log x$

$$\log_{10}1000 = \log_{10}10^3 = 3\log_{10}10 = 3$$

$$\log\sqrt{10} = \log(10)^{\frac{1}{2}} = \frac{1}{2}\log 10 = \frac{1}{2}\times 1 = \frac{1}{2}$$

The Change of Base Law

$$\log_a N = \frac{\log_b N}{\log_b a}$$

This Law is useful for change a logarithm in any base to a logarithm with a base such as 10 or e so that a calculator may be used.

Proof:

Let $m = \log_a N$

Using the definition of a Log: $N = a^m$

$$N = a^m$$

$\log_b N = \log_b a^m$ •take Log (base b) of both sides

$\log_b N = m\log_b a$ •using third law

$\frac{\log_b N}{\log_b a} = m$ •rearranging

$$\frac{\log_b N}{\log_b a} = \log_a N$$

Example

Evaluate $\log_3 7$

$$\log_3 7 = \frac{\log 7}{\log 3}\left(\text{or alternatively } \frac{\ln 7}{\ln 3}\right)$$

Using a calculator gives 1.7712 (to 4 d.p.)

Evaluate $\log_2 5 + \log_3 7$

$$\log_2 5 + \log_3 7$$
$$= \frac{\log 5}{\log 2} + \frac{\log 7}{\log 3}$$
$$= 4.0932\left(\text{to 4 d.p.}\right)$$

Video ‘Using Logarithm Laws’

Mixed examples:

Evaluate $\log_2 \frac{1}{64}$

$$\begin{aligned}&\log_2 \frac{1}{64}\\&= \log_2 2^{-6}\\&= -6\log_2 2\\&= -6\end{aligned}$$

- $\frac{1}{64} = \frac{1}{2^6} = 2^{-6}$

or

$$\begin{aligned}&\log_2 \frac{1}{64}\\&= \log_2 1 - \log_2 64\\&= 0 - \log_2 2^6\\&= 0 - 6\log_2 2\\&= 0 - 6\\&= -6\end{aligned}$$

Evaluate $3\log_3 25 - 2\log_3 5$

$$\begin{aligned}&3\log_3 25 - 2\log_3 5\\&= \log_3 25^3 - \log_3 5^2\\&= \log_3 \frac{25^3}{5^2}\\&= \log_3 \frac{5^6}{5^2}\\&= \log_3 5^4\\&= \log_3 625\end{aligned}$$

- $25^3 = \left(5^2\right)^3 = 5^6$

Evaluate $\frac{2\log_7 125}{3\log_7 5}$

$$\begin{aligned}&\frac{2\log_7 125}{3\log_7 5}\\&= \frac{2\log_7 5^3}{3\log_7 5}\\&= \frac{6\cancel{\log_7 5}}{3\cancel{\log_7 5}}\\&= 2\end{aligned}$$

Evaluate $\log_2 3 \times \log_3 4 \times \log_4 5 \times \log_5 8$

$$\begin{aligned}
&\log_2 3 \times \log_3 4 \times \log_4 5 \times \log_5 8 \\
&= \frac{\cancel{\log 3}}{\log 2} \times \frac{\cancel{\log 4}}{\cancel{\log 3}} \times \frac{\cancel{\log 5}}{\cancel{\log 4}} \times \frac{\log 8}{\cancel{\log 5}} \\
&= \frac{\log 8}{\log 2} \\
&= \frac{\log 2^3}{\log 2} \\
&= \frac{3\cancel{\log 2}}{\cancel{\log 2}} \\
&= 3
\end{aligned}$$

Write as a single logarithm $\log x + 3\log y - 4\log z$

$$\begin{aligned}
&\log x + 3\log y - 4\log z \\
&= \log x + \log y^3 - \log z^4 \\
&= \log \frac{xy^3}{z^4}
\end{aligned}$$

Write in terms of individual logs: $\ln \frac{x^2\sqrt{y}}{e}$

$$\begin{aligned}
&\ln \frac{x^2\sqrt{y}}{e} \\
&= \ln x^2 + \ln\sqrt{y} - \ln e \\
&= 2\ln x + \ln y^{\frac{1}{2}} - 1 \qquad \bullet\ \ln e = \log_e e = 1 \\
&= 2\ln x + \frac{1}{2}\ln y - 1
\end{aligned}$$

Simplify $2 + 4\log_3 x$

$$\begin{aligned}
&2 + 4\log_3 x && \bullet \text{ Express 2 as a Log with base 3} \\
&= \log_3 9 + \log_3 x^4 && \bullet\ 2 = 2\log_3 3 = \log_3 3^2 = \log_3 9 \\
&= \log_3 9x^4
\end{aligned}$$

Activity

1. Evaluate the following without a calculator

(a)	$\log_2 32$	(b)	$\log_5 125$
(c)	$\log_2 \frac{1}{16}$	(d)	$\log_{10} 0.001$
(e)	$\log_5 \sqrt{5}$	(f)	$\log_{\sqrt{2}} 2$
(g)	$\log_n \sqrt[4]{n}$	(h)	$\log_x \sqrt{x^3}$

2. Evaluate the following:

(a)	$\log_3 5 + \log_3 1.8$	(b)	$\log_5 10 + \log_5 50 + \log_5 0.25$
(c)	$\log_2 3 + \log_2 24 - \log_2 9$	(d)	$\log_7 8 - \log_7 6 - \log_7 4 + \log_7 3$
(e)	$\log_2 \sqrt{5} - \log_2 \sqrt{10}$	(f)	$2\log_2 3 - \log_2 36$
(g)	$5\log_3 8 - 2\log_3 16$	(h)	$3\log_{10} 5 - \log_{10} 2$
(i)	$\frac{1}{3}\log_2 27 - \frac{1}{2}\log_2 36$	(j)	$\frac{\log_3 36}{\log_3 6}$
(k)	$\frac{2\log_5 8}{3\log_5 4}$	(l)	$\frac{\log_7 \sqrt{x}}{\log_7 x\sqrt{x}}$
(m)	$\log_2 3 \times \log_3 4$	(n)	$\log_5 10 \times \log_{100} 5$

3. Write as a single Logarithm:

(a)	$\log_x a + \log_x b - \log_x c$	(b)	$2\log_b x + \log_b \sqrt{y} + \log_b 7 - 3\log_b z$
(c)	$2\log_a x - 2\log_a y - 2\log_a z$	(d)	$1 + 2\log_a b - \log_a ab$
(e)	$\log_3 abc + \log_3 b - 2\log_3 b$	(f)	$2\log_a (x+3) - \log_a (x^2 - 9)$
(g)	$2\log_a a + 2$	(h)	$\ln(x-5) + 2$

4. Write in terms of individual logs

(a) $\log_x ab^2c$ (b) $\log_a\left(\frac{x^2y}{\sqrt{z}}\right)$

(c) $\log_5\left(\sqrt[3]{x}y^2\right)$ (d) $\log_a\sqrt{xy^3}$

(e) $\log\left(\frac{1}{a}\right)$ (f) $\log_2\left(\frac{8}{ab^2}\right)$

5. Write in terms of individual logs

(a) $3\log 2x - 1$ (b) $3 - 2\log_3 x$

(c) $\log_5 x - 2$ (d) $2\log_2 x + 3\log_2 y + 2$

Solving Logarithms and Exponential Equations

Logarithmic Equations

There are two major ideas required when solving Logarithmic Equations.

The first is the Definition of a Logarithm. You may recall from an earlier topic:

> If a number (N) is written as a power with a base (b) and an exponent (e), such as
> $$N = b^e$$
> then $\log_b N = e$.
> The base, *b*, must be a positive number ($b > 0$) and not equal to 1 ($b \neq 1$).
> The number, *N*, must also be positive ($N > 0$)

For the equation below, changing the Logarithm form to Exponential form will solve for the variable *x*. (These were covered in the Definition of a Log topic)

$$\log_{10} x = 2$$
$$x = 10^2$$
$$x = 100$$

> Note:
>
> When solving any Logarithm equations, you should perform a check to see that the answer(s) obtained were consistent with the definition of a Log.

Similar to this, the question below is solved using the definition of a Log and knowledge of equations containing exponents.

$$\log_x 25 = 2$$
$$25 = x^2$$
$$x = \pm 5$$

Check:

As the base of a Log must be *(b > 0)* and *(b ≠ 1),* the solution *x=-5* is not consistent with the definition.

The solution to this question is *x=5*

This type of question can become more complex depending on the numbers. In this example, a calculator is required. You may need to review Exponents – Equations with Exponents.

$$\log_x 22 = 2.1$$
$$22 = x^{2.1}$$
$$x = \sqrt[2.1]{22}$$
$$x = 4.358 \text{ to 3 d.p.}$$

A third type is where the variable becomes the exponent. For example:
$\log_4 5 = x$ can be rewritten using the definition of a log to be $5 = 4^x$ which can be solved as an exponential equation (see below). However, a much easier method of solution is to use the Change of Base Rule.

$$\log_4 5 = x$$
$$x = \frac{\log 5}{\log 4} \quad \left(\text{alternatively } \frac{\ln 5}{\ln 4}\right)$$
$$x = 1.161 \text{ to 3 d.p.}$$

The second major idea is based on equivalence. Put simply:

If ....

$$\log_b M = \log_b N$$
$$\text{then}\ldots$$
$$M = N$$

Note: There must be a single Log term on each side of the equals sign.

Example:

$$\log x + \log 5 = \log 20$$
$$\log 5x = \log 20 \quad \bullet\text{Using Log Rule 1 (Product Rule)}$$
$$5x = 20$$
$$x = 4$$

Example:

$$2\ln x = \ln 36$$
$$\ln x^2 = \ln 36 \quad \bullet\text{Using Rule 3 (Power Rule)}$$
$$x^2 = 36$$
$$x = \pm 6$$

Check: The logarithm can only be found of positive values only, so the solution is *x=6*.

Some equations will contain a mix of Log terms and numbers, there are two strategies to solve this;

(i) Rearrange to get Log terms on one side and number(s) on the other,

or

(ii) Replace the number with an equivalent Log. For example - 1 is the same as $\log_4 4$, $\log_5 5$, $\log_{10} 10$ etc.

Example: Solve for *x*: $\log_4(2x+4)-2=\log_4 3$

Using strategy (i)		Using strategy (ii)
$\log_4(2x+4)-2=\log_4 3$		$\log_4(2x+4)-2=\log_4 3$
$\log_4(2x+4)-\log_4 3=2$		$\log_4(2x+4)-2\log_4 4=\log_4 3$
$\log_4\left(\frac{2x+4}{3}\right)=2$		$\log_4(2x+4)-\log_4 4^2=\log_4 3$
$\left(\frac{2x+4}{3}\right)=4^2$	or	$\log_4\left(\frac{2x+4}{16}\right)=\log_4 3$
$\frac{2x+4}{3}=16$		$\frac{2x+4}{16}=3$
$2x+4=48$		$2x+4=48$
$2x=44$		$2x=44$
$x=22$		$x=22$

Mixed examples:

(i) Solve for *x*. $\log_9 x=0.5$

$$\log_9 x=0.5$$
$$x=9^{0.5}$$
$$x=\sqrt{9}$$
$$x=3$$

(ii) Express *y* in terms of *x*: $\log y=2\log x+\log 8$

$$\log y=2\log x+\log 8$$
$$\log y=\log x^2+\log 8$$
$$\log y=\log 8x^2$$
$$y=8x^2$$

(iii) Express *y* in terms of *x*: $2\ln y=3+4\ln x$

$$2\ln y = 3 + 4\ln x$$
$$\ln y^2 - \ln x^4 = 3$$
$$\ln \frac{y^2}{x^4} = 3$$
$$\frac{y^2}{x^4} = e^3$$
$$y^2 = e^3 x^4$$
$$y = \sqrt{e^3 x^4}$$

(iv) Solve for x: $\log_6 x + \log_6 (x+5) = 2$

$$\log_6 x + \log_6 (x+5) = 2$$
$$\log_6 (x^2 + 5x) = 2$$
$$x^2 + 5x = 36$$
$$x^2 + 5x - 36 = 0$$
$$(x-4)(x+9) = 0$$
$$x = 4 \text{ or } -9$$

Check: Because it is not possible to have the Log of a negative number, the solution is $x = 4$.

Video 'Solving Logarithmic Equations'

Activity

1. Solve the following Logarithmic Equations.

(a)	$\log_{10} x = 2$	(b)	$\log_5 x = -2$
(c)	$\log_4 x = 0.5$	(d)	$\log_7 2x = 2$
(e)	$\log_2(x+4) = 3$	(f)	$\log_9\left(\frac{x}{4}\right) = -0.5$
(g)	$\log_3 x^2 = 2$	(h)	$\log_3 x = 2$
(i)	$\log_2(x^2+2x) = 3$	(j)	$\log_x 4 = 4$
(k)	$\log_x 9 = 0.5$	(l)	$\log_x 64 = 2$
(m)	$\log_x(2x-1) = 2$	(n)	$\log_x 3 = 2$
(o)	$\log_x 10 = 3$	(p)	$\log_4 64 = x$
(q)	$10^{\log x} = 4$	(r)	$e^{\ln(x+4)} = 7$

2. Solve for x in the Logarithmic Equations.

(a)	$2\log x = \log 4$	(b)	$2\ln 2x = \ln 36$
(c)	$\log(x+5) = \log x + \log 6$	(d)	$3\log x = \log 4$
(e)	$2\log x = \log 2x + \log 3$	(f)	$2\log_3 2 - \log_3(x+1) = \log_3 5$
(g)	$\log_4 x + \log_4 6 = \log_4 12$	(h)	$3\log_3 2 + \log_3(x-2) = \log_3 5$
(i)	$\log_8(x+2) = 2 - \log_8 2$	(j)	$\log_2 x = 5 - \log_2(x+4)$
(k)	$\log x + \log 3 = \log 5$	(l)	$\log x - \log(x-1) = \log 4$
(m)	$\log(x-3) = 3$	(n)	$\ln(4-x) + \ln 2 = 2\ln x$
(o)	$\log_4(2x+4) - 3 = \log_4 3$	(p)	$\log_2 x + 3\log_2 2 = \log_2 \frac{2}{x}$

3. Solve for y in terms of the other variables present.

(a)	$\log y = \log x + \log 4$	(b)	$\log y = \frac{1}{2}\log 5 + \frac{1}{2}\log x$
(c)	$\log y = \log 4 - \log x$	(d)	$\log y + \log x = \log 4 + 2$
(e)	$\log_5 x + \log_5 y = \log_5 3 + 1$	(f)	$2\ln y = \ln e^2 - 3\ln x$
(g)	$\log_2 y + 2\log_2 x = 5 - \log_2 4$	(h)	$\log_7 y = 2\log_7 5 + \log_7 x + 2$
(i)	$3\ln y = 2 + 3\ln x$	(j)	$2(\log_4 y - 3\log_4 x) = 3$
(k)	$2^y = e^x$	(l)	$10^y = 3^{x+1}$

Exponential Equations

An equation where the exponent (index) is a variable or contains a variable is called an exponential equation. An example is $3^x = 20$.

Let's consider the example below:

If an investor deposits $20 000 in an account that compounds at 5% p.a., how long is it before this amount has increased to $30 000? (Compounded annually)

Using the compound interest formula gives:

$$A = P(1+i)^n$$

$$30000 = 20000(1.05)^n$$

The first step is to rearrange the equation to get the form $a = b^x$ or $b^x = a$

$$\frac{30000}{20000} = 1.05^n$$

$$1.5 = 1.05^n$$

Then take the log of both sides. Because calculations will need to be performed, ordinary logarithms or natural logarithms should be used.

$1.5 = 1.05^n$	$1.5 = 1.05^n$
$\log 1.5 = \log 1.05^n$ taking the log of both sides	$\ln 1.5 = \ln 1.05^n$ taking the log of both sides
$\log 1.5 = n \log 1.05$ using the third log law	$\ln 1.5 = n \ln 1.05$ using the third log law
$\frac{\log 1.5}{\log 1.05} = n$ rearranging	$\frac{\ln 1.5}{\ln 1.05} = n$ rearranging
$n = 8.31$	$n = 8.31$

This means that it will take 8.31years (9 years in practice) for the growth in value to occur.

This can be checked by calculating:

$$1.05^{8.31}$$

$$= 1.5$$

This example contains a simple power.

$$3^x = 10$$

$$\log 3^x = \log 10$$

$$x \log 3 = \log 10$$

$$x = \frac{\log 10}{\log 3}$$

$$x = \frac{1}{\log 3}$$

$$x = 2.096$$

In this example, the 4 must be rearranged to obtain the power as the subject before taking the logarithm of both sides.

$$
\begin{aligned}
4 \times 7^x &= 20 \\
7^x &= \frac{20}{4} \\
7^x &= 5 \\
\log 7^x &= \log 5 \\
x \log 7 &= \log 5 \\
x &= \frac{\log 5}{\log 7} \\
x &= 0.8271
\end{aligned}
$$

In this example, there is a power as the subject; however, the exponent is more complex than previous examples.

$$
\begin{aligned}
12.5^{2n-1} &= 523.95 \\
\log 12.5^{2n-1} &= \log 523.95 \\
(2n-1)\log 12.5 &= \log 523.95 \\
2n-1 &= \frac{\log 523.95}{\log 12.5} \\
2n-1 &= 2.479 \\
2n &= 3.4790 \\
n &= 1.7395
\end{aligned}
$$

Example: Joan wants to retire when her superannuation fund reaches $500 000. She invests $1500 a month (after tax) into her superannuation fund. She assumes that the superannuation fund will return 6%pa or 0.5%pm. How long before her goal is reached?

This is a Future Value of an Annuity, with r = 0.005, R = 1500 and S = 500000.

$$
\begin{aligned}
S &= R\frac{\left[(1+r)^n - 1\right]}{r} \\
500000 &= 1500\frac{\left[(1+0.005)^n - 1\right]}{0.005} \\
2500 &= 1500\left[(1+0.005)^n - 1\right] && \bullet\text{mult both sides by 0.005} \\
1.666\overline{6} &= 1.005^n - 1 && \bullet\text{divide both sides by 1500} \\
2.666\overline{6} &= 1.005^n && \bullet\text{add 1 to both sides}
\end{aligned}
$$

To solve this exponential equation, logs are required.

$$2.666\overline{6} = 1.005^n$$

$$\log 2.666\overline{6} = \log 1.005^n \quad \text{•take Log of both sides}$$

$$\log 2.666\overline{6} = n \log 1.005 \quad \text{•use Log Law 3}$$

$$\frac{\log 2.666\overline{6}}{\log 1.005} = n$$

$$n = 197 \text{ (rounded up)}$$

It will be 197 months or 16 years 5 months before Joan has enough money.

Video ‘Solving Exponential Equations’

Activity

1. Solve the following exponential equations.

(a) $4^x = 20$

(b) $1.6^x = 3.2$

(c) $3^{2x} = 0.125$

(d) $4.6^{0.5x} = 100$

(e) $6^{x+1} = 40$

(f) $11.2^{2-x} = 0.6$

(g) $2^x + 12 = 40$

(h) $3 \times 6^x - 10 = 12.5$

(i) $6^{x+4} = 6^{11}$

(j) $5^x = 2^{2x+1}$

2. If an investor deposits $250 000 in an account that compounds at 7% p.a., how long is it before this amount has increased to $400 000? (Compounded semi-annually). Use the compound interest formula: $A = P(1+i)^n$

Answers to activity questions

Topic: Definition of a Log

1. Write the following in logarithmic form.

(a)
$$25 = 5^2$$
$$\log_5 25 = 2$$

(b)
$$1024 = 4^5$$
$$\log_4 1024 = 5$$

(c)
$$11.18 = 5^{1.5}$$
$$\log_5 11.18 = 1.5$$

(d)
$$908.14 = 3^{6.2}$$
$$\log_3 908.14 = 6.2$$

(e)
$$0.015625 = 8^{-2}$$
$$\log_8 0.015625 = -2$$

(f)
$$49^{0.5} = 7$$
$$7 = 49^{0.5}$$
$$\log_{49} 7 = 0.5$$

(g)
$$0.4884 = 6^{-0.4}$$
$$\log_6 0.4884 = -0.4$$

(h)
$$0.5 = 4^{-0.5}$$
$$\log_4 0.5 = -0.5$$

(i)
$$7.389 = e^2$$
$$\log_e 7.389 = 2 \ \ (\log_e \text{ means } \ln)$$
$$\ln 7.389 = 2$$

(j)
$$1.568 = e^{0.45}$$
$$\ln 1.568 = 0.45$$

2. Write the following in exponential form.

(a)
$$\log 100 = 2$$
$$\log_{10} 100 = 2$$
$$100 = 10^2$$

(b)
$$\log_2 1024 = 10$$
$$1024 = 2^{10}$$

(c)
$$\log 2.65 = 0.4232$$
$$2.65 = 10^{0.4232}$$

(d)
$$\log_4 100 = 3.322$$
$$100 = 4^{3.322}$$

(e)
$$\ln 1 = 0$$
$$1 = e^0$$

(f)
$$\ln 20 = 2.996$$
$$20 = e^{2.996}$$

(g)
$$\log_{100} 10 = \frac{1}{2}$$
$$10 = 100^{\frac{1}{2}}$$

(h)
$$\ln 5 = 1.6094$$
$$5 = e^{1.6094}$$

3. Evaluate the following logarithms using your calculator. Answer to 4 decimal places.

(a)
$$\log 250$$
[log] [2] [5] [0] [=] 2.3979

(b)
$$\log 2.95$$
[log] [2] [•] [9] [5] [=] 0.4698

(c) $\log 0.1$

[log] [0] [•] [1] [=] -1

(d) $\log\left(\frac{3}{8}\right)$

[log] [(] [3] [÷] [8] [)] [=] -0.4260

(e) $\ln 3.05$

[ln] [3] [•] [0] [5] [=] 1.1115

(f) $\ln 1000$

[ln] [1] [0] [0] [0] [=] 6.9078

(g) $\ln 0.025$

[ln] [0] [•] [0] [2] [5] [=] -3.6889

(h) $\ln\left(\frac{1}{9}\right)$

[ln] [(] [1] [÷] [9] [)] [=] -2.1972

(i)
$$\log_4 32$$
$$=\frac{\log 32}{\log 4}$$
$$=\frac{1.5051}{0.6021}$$
$$=2.5$$

(j)
$$\log_{0.25} 4$$
$$=\frac{\log 4}{\log 0.25}$$

[log] [4] [÷] [log] [0] [•] [2] [5] [=] -1

4. Working in reverse, find the value of x.

(a)
$$\log x = 0.7782$$
$$x = 10^{0.7782}$$
$$x = 6$$

(b)
$$\log x = 2.8779$$
$$x = 10^{2.8779}$$
$$x = 754.9$$

(c)
$$\log x = -0.9031$$
$$x = 10^{-0.9031}$$
$$x = 0.125$$

(d)
$$\log(2x-1) = 1.6902$$
$$2x-1 = 10^{1.6902}$$
$$2x-1 = 49 \quad \text{add 1 to both sides}$$
$$2x = 50 \quad \text{divide both sides by 2}$$
$$x = 25$$

(e)
$$\ln x = 2.72$$
$$x = e^{2.72}$$
$$x = 15.18$$

(f)
$$\ln x = 0.75$$
$$x = e^{0.75}$$
$$x = 2.117$$

(g)
$$\ln x = -0.5$$
$$x = e^{-0.5}$$
$$x = 0.6065$$

(h)
$$\ln 2x = \frac{3}{4}$$
$$\ln 2x = 0.75$$
$$2x = e^{0.75}$$
$$2x = 2.117$$
$$x = 1.0585$$

(i)
$$\log_2 x = 2.72$$
$$x = 2^{2.72}$$
$$x = 6.589$$

(j)
$$\log_5 x = -0.25$$
$$x = 5^{-0.25}$$
$$x = 0.6687$$

Topic: The Logarithm Laws

1. Evaluate the following without a calculator

(a)

$$\log_2 32$$
$$= \log_2 2^5$$
$$= 5\log_2 2$$
$$= 5$$

(b)

$$\log_5 125$$
$$= \log_5 5^3$$
$$= 3\log_5 5$$
$$= 3$$

(c)

$$\log_2 \frac{1}{16} \qquad \because \frac{1}{16} = \frac{1}{2^4} = 2^{-4}$$
$$= \log_2 2^{-4}$$
$$= -4$$

(d)

$$\log_{10} 0.001$$
$$= \log_{10} 10^{-3}$$
$$= -3\log_{10} 10$$
$$= -3$$

(e)

$$\log_5 \sqrt{5}$$
$$= \log_5 5^{\frac{1}{2}}$$
$$= \frac{1}{2}\log_5 5$$
$$= \frac{1}{2}$$

(f)

$$\log_{\sqrt{2}} 2$$
$$= \log_{\sqrt{2}} \left(\sqrt{2}\right)^2$$
$$= 2\log_{\sqrt{2}} \left(\sqrt{2}\right)$$
$$= 2$$

(g)

$$\log_n \sqrt[4]{n}$$
$$= \log_n n^{\frac{1}{4}}$$
$$= \frac{1}{4}\log_n n$$
$$= \frac{1}{4}$$

(h)

$$\log_x \sqrt{x^3} \qquad \because \sqrt{x^3} = \left(x^3\right)^{\frac{1}{2}} = x^{\frac{3}{2}}$$
$$= \log_x x^{\frac{3}{2}}$$
$$= \frac{3}{2}\log_x x$$
$$= \frac{3}{2}$$

2. Evaluate the following:

(a)

$$\log_3 5 + \log_3 1.8$$
$$= \log_3 5 \times 1.8$$
$$= \log_3 9 \qquad \because \log_3 9 = \log_3 3^2 = 2\log_3 3$$
$$= 2$$

(b)

$$\log_5 10 + \log_5 50 + \log_5 0.25$$
$$= \log_5 (10 \times 50 \times 0.25)$$
$$= \log_5 125$$
$$= 3$$

(c)

$$\log_2 3 + \log_2 24 - \log_2 9$$
$$= \log_2 \frac{(3\times 24)}{9}$$
$$= \log_2 8$$
$$= 3$$

(d)

$$\log_7 8 - \log_7 6 - \log_7 4 + \log_7 3$$
$$= \log_7 \frac{(8\times 3)}{(6\times 4)}$$
$$= \log_7 \frac{24}{24}$$
$$= \log_7 1$$
$$= 0$$

(e)

$$\log_2\sqrt{5}-\log_2\sqrt{10}$$
$$=\log_2\frac{\sqrt{5}}{\sqrt{10}}$$
$$=\log_2\frac{\cancel{\sqrt{5}}}{\sqrt{2}\times\cancel{\sqrt{5}}}$$
$$=\log_2\frac{1}{\sqrt{2}}$$
$$=\log_2 2^{-\frac{1}{2}}$$
$$=-\frac{1}{2}$$

(f)

$$2\log_2 3-\log_2 36$$
$$=\log_2\frac{3^2}{36}$$
$$=\log_2\frac{1}{4}$$
$$=\log_2 2^{-2}$$
$$=-2$$

(g)

$$5\log_3 8-2\log_3 16$$
$$=\log_3\frac{8^5}{16^2}$$
$$=\log_3\frac{2^{15}}{2^8}$$
$$=\log_3 2^7$$
$$=7\log_3 2$$

(h)

$$3\log_{10}5-\log_{10}2$$
$$=\log_{10}\frac{5^3}{2}$$
$$=\log_{10}62.5$$

(i)

$$\frac{1}{3}\log_2 27-\frac{1}{2}\log_2 36$$
$$=\log_2 27^{\frac{1}{3}}-\log_2 36^{\frac{1}{2}}$$
$$=\log_2 3-\log_2 6$$
$$=\log_2\frac{3}{6}$$
$$=\log_2 2^{-1}$$
$$=-1$$

(j)

$$\frac{\log_3 36}{\log_3 6}$$
$$=\frac{\log_3 6^2}{\log_3 6}$$
$$=\frac{2\cancel{\log_3 6}}{\cancel{\log_3 6}}$$
$$=2$$

(k)

$$\frac{2\log_5 8}{3\log_5 4}$$
$$=\frac{2\log_5 2^3}{3\log_5 2^2}$$
$$=\frac{6\log_5 2}{6\log_5 2}$$
$$=1$$

(l)

$$\frac{\log_7\sqrt{x}}{\log_7 x\sqrt{x}}$$
$$=\frac{\log_7 x^{\frac{1}{2}}}{\log_7 x^{\frac{3}{2}}}$$
$$=\frac{\frac{1}{2}\cancel{\log_7 x}}{\frac{3}{2}\cancel{\log_7 x}}$$
$$=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$$

(m)

$$\log_2 3\times\log_3 4 = \frac{\log 3}{\log 2}\times\frac{\log 4}{\log 3} = \frac{\log 4}{\log 2} = \frac{\log 2^2}{\log 2} = \frac{2\cancel{\log 2}}{\cancel{\log 2}} = 2$$

(n)

$$\log_5 10\times\log_{100} 5 = \frac{\log 10}{\cancel{\log 5}}\times\frac{\cancel{\log 5}}{\log 100} = \frac{1}{2}$$

3. Write as a single Logarithm:

(a)

$$\log_x a+\log_x b-\log_x c = \log_x \frac{ab}{c}$$

(b)

$$2\log_b x+\log_b \sqrt{y}+\log_b 7-3\log_b z = \log_b x^2+\log_b y^{\frac{1}{2}}+\log_b 7-\log_b z^3 = \log_b \frac{7x^2y^{\frac{1}{2}}}{z^3}$$

(c)

$$2\log_a x-2\log_a y-2\log_a z = 2(\log_a x-\log_a y-\log_a z) = 2\log_a \frac{x}{yz}$$

(d)

$$1+2\log_a b-\log_a ab = \log_a a+2\log_a b-(\log_a a+\log_a b) = \log_a a+2\log_a b-\log_a a-\log_a b = \log_a b$$

(e)

$$\log_3 abc+\log_3 b-2\log_3 b = \log_3 \frac{a\cancel{b^2}c}{\cancel{b^2}} = \log_3 ac$$

(f)

$$2\log_a(x+3)-\log_a(x^2-9) = \log_a(x+3)^2-\log_a(x^2-9) = \log_a \frac{(x+3)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \log_a \frac{(x+3)}{(x-3)}$$

(g)

$$2\log_a a+2 = 4$$

(h)

$$\ln(x-5)+2 = \ln(x-5)+2\ln e = \ln(x-5)+\ln e^2 = \ln e^2(x-5)$$

4. Write in terms of individual logs

(a)

$$\log_x ab^2c = \log_x a+\log_x b^2+\log_x c = \log_x a+2\log_x b+\log_x c$$

(b)

$$\log_a\left(\frac{x^2y}{\sqrt{z}}\right) = \log_x x^2+\log_x y-\log_x \sqrt{z} = 2\log_x x+\log_x y-\frac{1}{2}\log_x z = 2+\log_x y-\frac{1}{2}\log_x z$$

(c)

$$\log_5\left(\sqrt[3]{x}\,y^2\right)$$
$$= \log_5 \sqrt[3]{x} + \log_5 y^2$$
$$= \frac{1}{3}\log_5 x + 2\log_5 y$$

(d)

$$\log_a \sqrt{xy^3}$$
$$= \log_a (xy^3)^{\frac{1}{2}}$$
$$= \frac{1}{2}\log_a (xy^3)$$
$$= \frac{1}{2}\log_a (x) + \frac{1}{2}\log_a (y^3)$$
$$= \frac{1}{2}\log_a (x) + \frac{3}{2}\log_a (y)$$

(e)

$$\log\left(\frac{1}{a}\right)$$
$$= \log 1 - \log a$$
$$= -\log a$$

(f)

$$\log_2\left(\frac{8}{ab^2}\right)$$
$$= \log_2 8 - \log_2 (ab^2)$$
$$= \log_2 2^3 - (\log_2 a + \log_2 b^2)$$
$$= 3\log_2 2 - \log_2 a - 2\log_2 b$$
$$= 3 - \log_2 a - 2\log_2 b$$

5. Write in terms of individual logs

(a)
$$3\log 2x-1$$
$$=\log(2x)^3-\log 10$$
$$=\log\frac{8x^3}{10}$$
$$=\log\frac{4x^3}{5}$$

(b)
$$3-2\log_3 x$$
$$=3\log_3 3-2\log_3 x$$
$$=\log_3 3^3-\log_3 x^2$$
$$=\log_3\frac{27}{x^2}$$

(c)
$$\log_5 x-2$$
$$=\log_5 x-2\log_5 5$$
$$=\log_5\left(\frac{x}{25}\right)$$

(d)
$$2\log_2 x+3\log_2 y+5$$
$$=\log_2 x^2+\log_2 y^3+\log_2 2^5$$
$$=\log_2\left(32x^2y^3\right)$$

Topic: Solving Logarithmic and Exponential Equations

1. Solve the following Logarithmic Equations.

(a)
$$\log_{10} x=2$$
$$x=10^2$$
$$x=100$$

(b)
$$\log_5 x=-2$$
$$x=5^{-2}$$
$$x=\frac{1}{25}$$

(c)
$$\log_4 x=0.5$$
$$x=4^{0.5}\quad \text{or } \sqrt{4}$$
$$x=2$$

(d)
$$\log_7 2x=2$$
$$2x=7^2$$
$$2x=49$$
$$x=\frac{49}{2}=24.5$$

(e)
$$\log_2(x+4)=3$$
$$x+4=2^3$$
$$x+4=8$$
$$x=4$$

(f)
$$\log_9\left(\frac{x}{4}\right)=-0.5$$
$$\frac{x}{4}=9^{-0.5}$$
$$\frac{x}{4}=\frac{1}{\sqrt{9}}\qquad 9^{0.5}=\sqrt{9}$$
$$\frac{x}{4}=\frac{1}{3}$$
$$x=\frac{4}{3}$$

(g)
$$\log_3 x^2=2$$
$$x^2=3^2$$
$$x^2=9$$
$$x=\pm\sqrt{9}=\pm 3$$

(h)
$$\log_3 x=2$$
$$x=3^2$$
$$x=9$$

(i)
$$\log_2\left(x^2+2x\right)=3$$
$$x^2+2x=2^3$$
$$x^2+2x-8=0$$
$$(x+4)(x-2)=0$$
$$x=-4 \text{ or } 2$$

Checking; when *x=-4*, $\log_2 8=3$

(j)
$$\log_x 4=4$$
$$4=x^4$$
$$x=\sqrt[4]{4}$$
$$x=1.4142 \text{ to 4 d.p.}$$

when $x=2$, $\log_2 8=3$

(k)
$$\log_x 9 = 0.5$$
$$9 = x^{0.5}$$
$$x = 9^2$$
$$x = 81$$

(l)
$$\log_x 64 = 2$$
$$64 = x^2$$
$$x = \pm\sqrt{64}$$
$$x = 8$$
Bases cannot be negative

(m)
$$\log_x(2x-1) = 2$$
$$2x-1 = x^2$$
$$0 = x^2 - 2x + 1$$
$$0 = (x-1)^2$$
$$x-1 = 0$$
$$x = 1$$
There are two answers of $x=1$

(n)
$$\log_x 3 = 2$$
$$3 = x^2$$
$$x = \pm\sqrt{3}$$
$$x = 1.732$$
Bases cannot be negative

(o)
$$\log_x 10 = 3$$
$$10 = x^3$$
$$x = \sqrt[3]{10}$$
$$x = 2.154$$

(p)
$$\log_4 64 = x$$
$$x = \frac{\log 64}{\log 4} \quad \text{change of base}$$
$$x = 3$$

(q)
$$10^{\log x} = 4$$
$$x = 4$$
or
$$10^{\log x} = 4$$
$$\log 4 = \log x \quad \text{Definition}$$
$$x = 4$$

(r)
$$e^{\ln(x+4)} = 7$$
$$x + 4 = 7$$
$$x = 3$$

2. Solve for x in the Logarithmic Equations.

(a)
$$2\log x = \log 4$$
$$\log x^2 = \log 4$$
$$x^2 = 4$$
$$x = 2 \text{ (-2 is not possible)}$$

(b)
$$2\ln 2x = \ln 36$$
$$\ln(2x)^2 = \ln 36$$
$$4x^2 = 36$$
$$x^2 = 9$$
$$x = 3 \text{ (-3 is not possible)}$$

(c)
$$\log(x+5) = \log x + \log 6$$
$$\log(x+5) = \log 6x$$
$$x+5 = 6x$$
$$5 = 5x$$
$$x = 1$$

(d)
$$3\log x = \log 4$$
$$\log x^3 = \log 4$$
$$x^3 = 4$$
$$x = \sqrt[3]{4} \approx 1.587 \text{ to 3 d.p.}$$

(e)
$$2\log x = \log 2x + \log 3$$
$$\log x^2 = \log 6x$$
$$x^2 = 6x$$
$$x^2 - 6x = 0$$
$$x(x-6) = 0$$
$$x = 6 \text{ (0 is not possible)}$$

(f)
$$2\log_3 2 - \log_3(x+1) = \log_3 5$$
$$\log_3 4 - \log_3(x+1) = \log_3 5$$
$$\log_3\left(\frac{4}{x+1}\right) = \log_3 5$$
$$\frac{4}{x+1} = 5$$
$$4 = 5x + 5$$
$$-1 = 5x$$
$$x = \frac{-1}{5}$$

(g)

$$\log_4 x + \log_4 6 = \log_4 12$$
$$\log_4 6x = \log_4 12$$
$$6x = 12$$
$$x = 2$$

(h)

$$3\log_3 2 + \log_3(x-2) = \log_3 5$$
$$\log_3 2^3 + \log_3(x-2) = \log_3 5$$
$$\log_3 8\left(x-2\right) = \log_3 5$$
$$8x - 16 = 5$$
$$8x = 21$$
$$x = \frac{21}{8} \text{ or } 2\frac{5}{8}$$

(i)

$$\log_8(x+2) = 2 - \log_8 2$$
$$\log_8(x+2) + \log_8 2 = 2$$
$$\log_8 2(x+2) = 2$$
$$2x + 4 = 64$$
$$2x = 60$$
$$x = 30$$

(j)

$$\log_2 x = 5 - \log_2(x+4)$$
$$\log_2 x + \log_2(x+4) = 5$$
$$\log_2 x(x+4) = 5$$
$$x^2 + 4x = 2^5$$
$$x^2 + 4x - 32 = 0$$
$$(x-4)(x+8) = 0$$
$$x = 4$$

(-8 is not possible)

(k)

$$\log x + \log 3 = \log 5$$
$$\log 3x = \log 5$$
$$3x = 5$$
$$x = \frac{5}{3}$$

(l)

$$\log x - \log(x-1) = \log 4$$
$$\log\left(\frac{x}{x-1}\right) = \log 4$$
$$\frac{x}{x-1} = 4$$
$$x = 4x - 4$$
$$4 = 3x$$
$$x = \frac{4}{3}$$

(m)

$$\log(x-3) = 3$$
$$x - 3 = 10^3$$
$$x - 3 = 1000$$
$$x = 1003$$

(n)

$$\ln(4-x) + \ln 2 = 2\ln x$$
$$\ln 2(4-x) = \ln x^2$$
$$8 - 2x = x^2$$
$$0 = x^2 + 2x - 8$$
$$0 = (x+4)(x-2)$$
$$x = 2 \text{ (-4 is not possible)}$$

(o)

$$\log_4(2x+4) - 3 = \log_4 3$$
$$\log_4(2x+4) - \log_4 3 = 3$$
$$\log_4\left(\frac{2x+4}{3}\right) = 3$$
$$\frac{2x+4}{3} = 64$$
$$2x + 4 = 192$$
$$2x = 188$$
$$x = 94$$

(p)

$$\log_2 x + 3\log_2 2 = \log_2 \frac{2}{x}$$
$$\log_2 x + \log_2 2^3 = \log_2 \frac{2}{x}$$
$$\log_2 8x = \log_2 \frac{2}{x}$$
$$8x = \frac{2}{x}$$
$$8x^2 = 2$$
$$x^2 = \frac{1}{4}$$
$$x = \frac{1}{2} \quad (-\frac{1}{2} \text{ is not possible})$$

3. Solve for y in terms of the other variables present.

(a)
$$\log y = \log x + \log 4$$
$$\log y = \log 4x$$
$$y = 4x$$

(b)
$$\log y = \frac{1}{2}\left[\log 5 + \log x\right]$$
$$\log y = \frac{1}{2}\log 5x$$
$$\log y = \log\sqrt{5x} \qquad \square(5x)^{\frac{1}{2}} = \sqrt{5x}$$
$$y = \sqrt{5x}$$

(c)
$$\log y = \log 4 - \log x$$
$$\log y = \log\frac{4}{x}$$
$$y = \frac{4}{x}$$

(d)
$$\log y + \log x = \log 4 + 2$$
$$\log xy - \log 4 = 2$$
$$\log\left(\frac{xy}{4}\right) = 2$$
$$\frac{xy}{4} = 10^2$$
$$xy = 400$$
$$y = \frac{400}{x}$$

(e)
$$\log_5 x + \log_5 y = \log_5 3 + 1$$
$$\log_5 x + \log_5 y - \log_5 3 = 1$$
$$\log_5\left(\frac{xy}{3}\right) = 1$$
$$\frac{xy}{3} = 5$$
$$xy = 15$$
$$y = \frac{15}{x}$$

(f)
$$2\ln y = \ln e^2 - 3\ln x$$
$$\ln y^2 = \ln e^2 - \ln x^3$$
$$\ln y^2 = \ln\frac{e^2}{x^3}$$
$$y^2 = \frac{e^2}{x^3}$$
$$y = \sqrt{\frac{e^2}{x^3}}$$

(g)
$$\log_2 y + 2\log_2 x = 5 - \log_2 4$$
$$\log_2 y + \log_2 x^2 = 5 - 2$$
$$\log_2 x^2 y = 3$$
$$x^2 y = 2^3$$
$$y = \frac{8}{x^2}$$

(h)
$$\log_7 y = 2\log_7 5 + \log_7 x + 2$$
$$\log_7 y - \log_7 5^2 - \log_7 x = 2$$
$$\log_7\frac{y}{25x} = 2$$
$$\frac{y}{25x} = 7^2$$
$$y = 1225x$$

(i)
$$3\ln y = 2 + 3\ln x$$
$$\ln y^3 = 2 + \ln x^3$$
$$\ln y^3 - \ln x^3 = 2$$
$$\ln\frac{y^3}{x^3} = 2$$
$$\frac{y^3}{x^3} = e^2$$
$$y^3 = e^2 x^3$$
$$y = \sqrt[3]{e^2 x^3}$$

(j)
$$2(\log_4 y - 3\log_4 x) = 3$$
$$2(\log_4 y - \log_4 x^3) = 3$$
$$\log_4\left(\frac{y}{x^3}\right)^2 = 3$$
$$\frac{y^2}{x^6} = 64$$
$$y^2 = 64x^6$$
$$y = 8x^3$$

(k)

$$2^y = e^x$$
$$\ln 2^y = x$$
$$y \ln 2 = x$$
$$y = \frac{x}{\ln 2}$$

(l)

$$10^y = 3^{x+1}$$
$$\log 10^y = \log 3^{x+1}$$
$$y \log 10 = \log 3^{x+1}$$
$$y = (x+1)\log 3$$

1. Solve the following exponential equations.

(a)

$$4^x = 20$$
$$\log 4^x = \log 20 \quad \text{Take log of both sides}$$
$$x\log 4 = \log 20 \quad \text{Using the third log law}$$
$$x = \frac{\log 20}{\log 4}$$
$$x = 2.161$$

or

$$4^x = 20$$
$$\ln 4^x = \ln 20 \quad \text{Take natural log of both sides}$$
$$x\ln 4 = \ln 20 \quad \text{Using the third log law}$$
$$x = \frac{\ln 20}{\ln 4}$$
$$x = 2.161$$

(b)

$$1.6^x = 3.2$$
$$\log 1.6^x = \log 3.2$$
$$x\log 1.6 = \log 3.2$$
$$x = \frac{\log 3.2}{\log 1.6}$$
$$x = 2.475$$

$$1.6^x = 3.2$$
$$\ln 1.6^x = \ln 3.2$$
$$x\ln 1.6 = \ln 3.2$$
$$x = \frac{\ln 3.2}{\ln 1.6}$$
$$x = 2.475$$

(c)

$$3^{2x} = 0.125$$
$$\log 3^{2x} = \log 0.125$$
$$2x\log 3 = \log 0.125$$
$$2x = \frac{\log 0.125}{\log 3}$$
$$2x = -1.8928$$
$$x = \frac{-1.8928}{2}$$
$$x = -0.946$$

$$3^{2x} = 0.125$$
$$\ln 3^{2x} = \ln 0.125$$
$$2x\ln 3 = \ln 0.125$$
$$2x = \frac{\ln 0.125}{\ln 3}$$
$$2x = -1.8928$$
$$x = \frac{-1.8928}{2}$$
$$x = -0.946$$

(d)

$$4.6^{0.5x} = 100$$
$$\log 4.6^{0.5x} = \log 100$$
$$0.5x\log 4.6 = \log 100$$
$$0.5x = \frac{\log 100}{\log 4.6}$$
$$0.5x = 3.018$$
$$x = \frac{3.0177}{0.5}$$
$$x = 6.035$$

$$4.6^{0.5x} = 100$$
$$\ln 4.6^{0.5x} = \ln 100$$
$$0.5x\ln 4.6 = \ln 100$$
$$0.5x = \frac{\ln 100}{\ln 4.6}$$
$$0.5x = 3.018$$
$$x = \frac{3.0177}{0.5}$$
$$x = 6.035$$

(e)

$$6^{x+1} = 40$$
$$\log 6^{x+1} = \log 40$$
$$(x+1)\log 6 = \log 40$$
$$x+1 = \frac{\log 40}{\log 6}$$
$$x+1 = 2.0588$$
$$x = 2.0588 - 1$$
$$x = 1.0588$$

$$6^{x+1} = 40$$
$$\ln 6^{x+1} = \ln 40$$
$$(x+1)\ln 6 = \ln 40$$
$$x+1 = \frac{\ln 40}{\ln 6}$$
$$x+1 = 2.0588$$
$$x = 2.0588 - 1$$
$$x = 1.0588$$

(f)

$$11.2^{2-x} = 0.6$$
$$\log 11.2^{2-x} = \log 0.6$$
$$(2-x)\log 11.2 = \log 0.6$$
$$2-x = \frac{\log 0.6}{\log 11.2}$$
$$2-x = -0.2114$$
$$2+0.2114 = x$$
$$x = 2.2114$$

$$11.2^{2-x} = 0.6$$
$$\ln 11.2^{2-x} = \ln 0.6$$
$$(2-x)\ln 11.2 = \ln 0.6$$
$$2-x = \frac{\ln 0.6}{\ln 11.2}$$
$$2-x = -0.2114$$
$$2+0.2114 = x$$
$$x = 2.2114$$

(g)

$2^x + 12 = 40$ rearrange to get 2^x as the subject

$$2^x = 40-12$$
$$2^x = 28$$
$$\log 2^x = \log 28$$
$$x\log 2 = \log 28$$
$$x = \frac{\log 28}{\log 2}$$
$$x = 4.8074$$

$2^x + 12 = 40$ rearrange to get 2^x as the subject

$$2^x = 40-12$$
$$2^x = 28$$
$$\ln 2^x = \ln 28$$
$$x\ln 2 = \ln 28$$
$$x = \frac{\ln 28}{\ln 2}$$
$$x = 4.8074$$

(h)

$3\times 6^x - 10 = 12.5$ rearrange to make 6^x the subject

$$3\times 6^x = 12.5+10$$
$$3\times 6^x = 22.5$$
$$6^x = \frac{22.5}{3}$$
$$6^x = 7.5$$
$$\log 6^x = \log 7.5$$
$$x\log 6 = \log 7.5$$
$$x = \frac{\log 7.5}{\log 6}$$
$$x = 1.1245$$

$3\times 6^x - 10 = 12.5$ rearrange to make 6^x the subject

$$3\times 6^x = 12.5+10$$
$$3\times 6^x = 22.5$$
$$6^x = \frac{22.5}{3}$$
$$6^x = 7.5$$
$$\ln 6^x = \ln 7.5$$
$$x\ln 6 = \ln 7.5$$
$$x = \frac{\ln 7.5}{\ln 6}$$
$$x = 1.1245$$

(i)

$$6^{x+4} = 6^{11}$$

as the bases are the same

the exponents can be equated

$$x+4 = 11$$
$$x = 7$$

(j)

$5^x = 2^{2x+1}$	$5^x = 2^{2x+1}$
$\log 5^x = \log 2^{2x+1}$	$\ln 5^x = \ln 2^{2x+1}$
$x\log 5 = (2x+1)\log 2$	$x\ln 5 = (2x+1)\ln 2$
$x = (2x+1)\dfrac{\log 2}{\log 5}$	$x = (2x+1)\dfrac{\ln 2}{\ln 5}$
$x = 0.4307(2x+1)$	$x = 0.4307(2x+1)$
$x = 0.8614x + 0.4307$	$x = 0.8614x + 0.4307$
$1x - 0.8614x = 0.4307$	$1x - 0.8614x = 0.4307$
$0.1386x = 0.4307$	$0.1386x = 0.4307$
$x = \dfrac{0.4307}{0.1386}$	$x = \dfrac{0.4307}{0.1386}$
$x = 3.108$	$x = 3.108$

2. If an investor deposits $250 000 in an account that compounds at 7% p.a., how long is it before this amount has increased to $400 000? (Compounded semi annually – half yearly). Use the compound interest formula: $A = P(1+i)^n$

7% p.a. = 3.5% per half year

$$A = P(1+i)^n$$

$$400000 = 250000(1+0.035)^n$$

$$\frac{400000}{250000} = 1.035^n$$

$$1.6 = 1.035^n$$

$$\log 1.6 = \log 1.035^n$$

$$\frac{\log 1.6}{\log 1.035} = n$$

$$n = 13.66$$

It will take 13.66 half years for $250 000 to grow to $400 000. This is 6.83 years or, in reality, 7 years to the end of the period.